Doppler Effect and "Supersonic" Motion

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The theory and method used for the animations are described. Here "supersonic" simply refers to the source moving at a speed faster than the wave speed.

1 Mathematical Formulation

Consider the wave equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi s(\boldsymbol{x},t), \tag{1}$$

where c is the wave speed and s is a source term. The equation can be solved using the method of Green's function. The retarded solution is given by (see, e.g., Ch. 6 of *Classical Electrodynamics* by J.D. Jackson)

$$\psi(\boldsymbol{x},t) = \int \frac{s(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|} d^3x', \tag{2}$$

where $t_r = t - |\boldsymbol{x}-\boldsymbol{x}'|/c$ is the retarded time. Consider a point source emitting a wave of constant frequency f at its rest frame at $t > 0$, and it is moving along a trajectory given by $\boldsymbol{x}_s(t)$. The source function can be written as

$$s(\boldsymbol{x},t) = A\delta(\boldsymbol{x}-\boldsymbol{x}_s(t))e^{-i\omega t}\Theta(t), \tag{3}$$

where $\omega = 2\pi f$, A is a constant related to the amplitude of the generated waves, $\Theta(t) = 0$ for $t < 0$ and 1 for $t > 0$ is the Heaviside step function. The function $\Theta(t)$ is inserted so that the source emits waves only after $t > 0$.

Substituting Eq. (3) into (2) yields the following Liénard-Wiechert-like formula:

$$\psi(\boldsymbol{x},t) = \frac{Ae^{-i\omega t_r}}{|\boldsymbol{x}-\boldsymbol{x}_s(t_r)|[1-\boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)]}\Theta(t_r), \tag{4}$$

where $\boldsymbol{\beta}_s(t_r) = \boldsymbol{v}_s(t_r)/c$, $\boldsymbol{v}_s(t_r) = \dot{\boldsymbol{x}}_s(t_r)$ is the velocity of the source at the retarded time, and $\hat{\boldsymbol{n}}(t_r) = (\boldsymbol{x}-\boldsymbol{x}_s(t_r))/|\boldsymbol{x}-\boldsymbol{x}_s(t_r)|$. Here we assume that there is only one retarded time t_r that satisfies the equation $c(t-t_r) = |\boldsymbol{x}-\boldsymbol{x}_s(t_r)|$. This is true if the source moves slower than the wave speed. If t_{r1} and t_{r2} are solutions of the equation and $t_{r2} > t_{r1}$, we have

$$c(t-t_{r1}) = |\boldsymbol{x}-\boldsymbol{x}_s(t_{r1})| \quad , \quad c(t-t_{r2}) = |\boldsymbol{x}-\boldsymbol{x}_s(t_{r2})|. \tag{5}$$

Subtracting the two equations yields

$$c(t_{r2}-t_{r1}) = |\boldsymbol{x}-\boldsymbol{x}_s(t_{r1})| - |\boldsymbol{x}-\boldsymbol{x}_s(t_{r2})|. \tag{6}$$

Since

$$|\boldsymbol{A}|-|\boldsymbol{B}| = \sqrt{(|\boldsymbol{A}|-|\boldsymbol{B}|)^2} = \sqrt{|\boldsymbol{A}|^2+|\boldsymbol{B}|^2-2|\boldsymbol{A}||\boldsymbol{B}|} \leq \sqrt{|\boldsymbol{A}|^2+|\boldsymbol{B}|^2-2\boldsymbol{A}\cdot\boldsymbol{B}} = |\boldsymbol{A}-\boldsymbol{B}|, \quad (7)$$

it follows that

$$c(t_{r2}-t_{r1}) \leq |\boldsymbol{x}_s(t_{r2})-\boldsymbol{x}_s(t_{r1})| \quad \Rightarrow \quad \frac{|\boldsymbol{x}_s(t_{r2})-\boldsymbol{x}_s(t_{r1})|}{t_{r2}-t_{r1}} \geq c. \quad (8)$$

This means that the average speed of the source in the time interval (t_{r1}, t_{r2}) is faster than the wave speed. So multiple retarded times can only occur if $v_s \geq c$. In the case of multiple retarded times, ψ is given by the sum of contributions from these retarded times.

A special case is when the source is stationary: $\boldsymbol{x}_s = 0$ at all time. Then $t_r = t - r/c$, where $r = |\boldsymbol{x}|$ is the distance from the source. Equation (4) becomes

$$\psi(\boldsymbol{x}, t) = \frac{Ae^{i(kr-\omega t)}}{r}\Theta\left(t - \frac{r}{c}\right), \quad (9)$$

where $k = \omega r/c$. This solution describes an outgoing spherical wave.

2 Animation Setup

The animation is based on Eq. (4). For a given point $\boldsymbol{x}$, the wave function ψ is obtained by finding the retarded position of the source and then apply Eq. (4). We can turn it around and consider a time sequence $t_1 < t_2 < \cdots < t_n$. At any given time $t \geq t_n$, a wave emitted at time t_i reaches a spherical surface of radius $c(t-t_i)$ centered at $\boldsymbol{x}_s(t_i)$. The retarded time for points on this sphere is t_i and the wave function of points on this sphere is $Ae^{-i\omega t_i}/[c(t-t_i)(1-\boldsymbol{\beta}(t_i)\cdot\hat{\boldsymbol{n}}(t_i))]$.

The simplest choice of the time sequence $\{t_i\}$ is $t_i = i/f$, $((i = 0, 1, 2, \cdots)$. In this case, the phase of the wave at these t_i are the same and the spheres $\{S_i : |\boldsymbol{x} - \boldsymbol{x}_s(t_i)| = c(t-t_i)\}$ are the wavefronts.

The procedure for creating the animation is very simple:

1. For any given time $t > 0$, calculate $n = [tf]$, where $[x]$ denotes the largest integer smaller than x. The value of n is the number of times t_i in the time sequence so far.

2. For each $i \in [0, n]$, draw a circle of radius $c(t-t_i)$ centered at $\boldsymbol{x}_s(t_i)$. A circle is drawn instead of a sphere to make the animation two-dimensional. The n circles represent the wavefronts.

3. Draw the source position at $\boldsymbol{x}_s(t)$.

An animation is produced by generating a new picture at different t every 20 ms. The procedure is intuitive. It simply says a wavefront generated at time t_i spreads out from the source at the wave speed. The distance between the successive wavefront is the wavelength. Since the wave speed is constant, the wavelength at a given point is inversely proportional to the observed frequency. The observed frequency at a given point can also be measured by counting the number of wavefronts passing through the point per unit time.

It is also easy to see that when the source speed $v_s < c$ at all times, the retarded time at any point is unique since a wave emitted at t_i is always inside a sphere of the wave generated at $t < t_i$, so the spheres never cross. However, if $v_s > c$, a wave emitted at t_i is outside the sphere of a wave generated at a slightly earlier time and the two spheres will cross as they expand. When that happens, the intersection points will have two different retarded times.

We consider two cases of $\boldsymbol{x}_s(t)$ in the animation page. The first case is a motion with constant velocity in which $\boldsymbol{x}_s(t) = \boldsymbol{v}t$. In the second case the source moves in a circle of radius r_s with constant speed v_s in which $\boldsymbol{x}_s(t) = r(\cos\Omega t\hat{\boldsymbol{x}} + \sin\Omega t\hat{\boldsymbol{y}})$. Here $\Omega = v_s/r_s$.

3 Doppler Effect: Moving Source vs Moving Observer

It follows from Eq. (4) that the phase of the wave is

$$\phi(\boldsymbol{x}, t) = \phi_A - \omega t_r(\boldsymbol{x}, t), \tag{10}$$

where ϕ_A is the phase of the constant A: $A = |A|e^{i\phi_A}$. The observed angular frequency is

$$\omega_{\rm obs} = \left|\frac{d\phi}{dt}\right| = \omega\frac{dt_r}{dt}. \tag{11}$$

Let $\boldsymbol{x}_{\rm obs}(t)$ be the observer's position as a function of t. Then $t_r = t - |\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)|/c$ and

$$\begin{aligned}
\frac{dt_r}{dt} &= 1 - \frac{1}{c}\frac{d}{dt}|\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)| && (12)\\
&= 1 - \frac{1}{c}\left[\dot{\boldsymbol{x}}_{\rm obs}(t)\cdot\boldsymbol{\nabla}_{\boldsymbol{x}_{\rm obs}}|\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)| + \frac{d\boldsymbol{x}_s(t_r)}{dt}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}_s}|\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)|\right] && (13)\\
&= 1 - \frac{1}{c}\left[\boldsymbol{v}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r) - \frac{dt_r}{dt}\dot{\boldsymbol{x}}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)\right] && (14)\\
&= 1 - \frac{1}{c}\left[\boldsymbol{v}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r) - \frac{dt_r}{dt}\boldsymbol{v}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)\right] && (15)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \left(1 - \frac{\boldsymbol{v}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)}{c}\right)\frac{dt_r}{dt} &= 1 - \frac{\boldsymbol{v}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r)}{c} && (16)\\
\Rightarrow\quad \frac{dt_r}{dt} &= \frac{1 - \boldsymbol{\beta}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r)}{1 - \boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)}, && (17)
\end{aligned}$$

where $\hat{\boldsymbol{n}}(t_r) = [\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)]/|\boldsymbol{x}_{\rm obs}(t) - \boldsymbol{x}_s(t_r)|$, $\boldsymbol{\beta}_{\rm obs} = \boldsymbol{v}_{\rm obs}/c$, $\boldsymbol{\beta}_s = \boldsymbol{v}_s/c$, and all velocities are measured relative to the rest frame of the medium in which the waves propogate. Combining Eqs. (17) and (11) yields

$$\omega_{\rm obs}(t) = \frac{1 - \boldsymbol{\beta}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r)}{1 - \boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)}\,\omega\ . \tag{18}$$

Equation (18) is a general formula of the Doppler shift when both the source and observer are moving relative to the medium. If the observer is stationary, $\boldsymbol{\beta}_{\rm obs} = 0$ and the equation reduces to

$$\omega_{\rm obs}(t) = \frac{\omega}{1 - \boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}(t_r)}\ . \tag{19}$$

For a stationary source, $\boldsymbol{\beta}_s = 0$ and the equation reduces to

$$\omega_{\rm obs}(t) = \omega[1 - \boldsymbol{\beta}_{\rm obs}(t)\cdot\hat{\boldsymbol{n}}(t_r)]. \tag{20}$$

These two formulae can also be derived easily using a geometric approach. The difference between a moving source and a moving observer is that for a moving source, the wave speed is the same in every direction but the wavelength depends on the direction: compressed in the direction of the source's motion and expanded in the opposite direction. In the case of a moving observer and stationary source, the wavelengths are the same everywhere but the wave speed is anisotropic in the observer's rest frame.

Although the two formulae both imply that the observed frequency increases (decreases) when the source is moving towards (away from) the observer, the amount of shift is slightly different. However, both formulae give the same result to first order in v/c when $v \ll c$. Note that this analysis cannot be applied to light since light does not propagate through a medium. The Doppler shift of light depends only on the relative velocity between the source and observer.

In the case of a stationary observer and moving source, the wave speed is unchanged. The change in frequency means that the wavelength changes. In particular, the wavelength is multiplied by a factor $1-\boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}$ compared to the wavelength emitted by a stationary source. On the other hand, equation (4) implies that the amplitude of the wave also changes as a result of the source's motion: the amplitude is multiplied by a factor $1/[1-\boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}}]$, which is exactly the inverse of the wavelength factor. This means that the density of the wavefront observed in an animation is proportional to the enhancement factor of the wave amplitude.

4 Motion with $v_s > c$

One important feature of $v_s > c$ motion is the presence of a shock wave. This can be seen from equation (4). When $v_s > c$, there is a direction $\hat{\boldsymbol{n}}$ in which $\boldsymbol{\beta}_s \cdot \hat{\boldsymbol{n}} = 1$ and ψ is singular in that direction. In many cases, the wave equation (1) arises from a linear perturbation analysis. When the amplitude of the wave becomes large, the wave equation is no longer valid and nonlinear effects must be taken into account. The shock structures for the constant velocity motion and circular motion are analyzed below.

4.1 Constant Velocity Motion

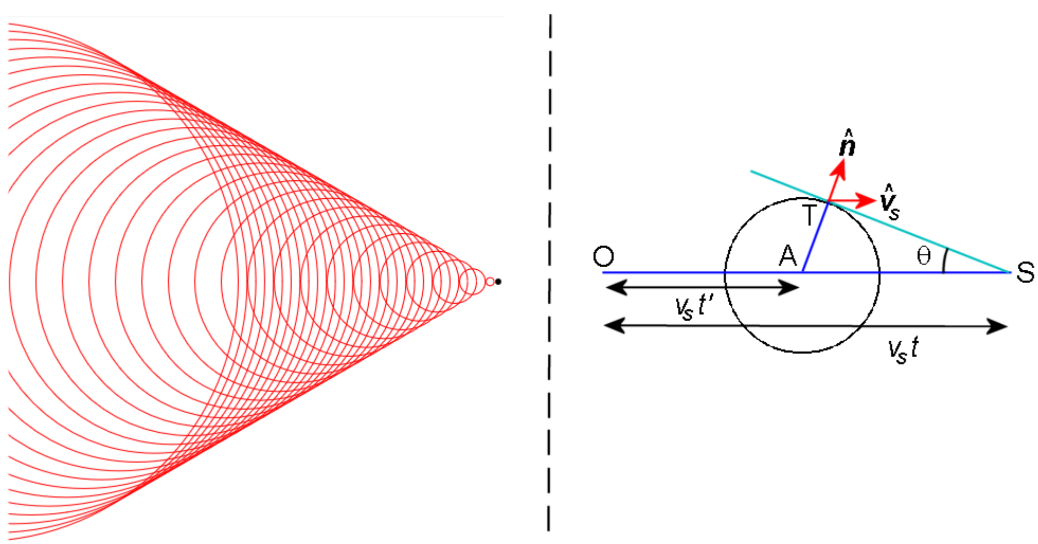

Figure 1: Left: Bow wave generated by a source (represented by a black dot) moving with constant velocity $v_s = 2c$ to the right. Red lines are wavefronts of waves generated at earlier times. Right: The geometry of the Mach cone.

For the constant-velocity motion, the shock wave forms the well-known Mach cone as shown on the left side of Figure 1. This is also known as a bow wave, often seen when a ship moves through the water. The geometry of the Mach cone is shown on the right side of Figure 1. At

$t=0$, the source is at point O. At time t, the source moves to point S, which is at a distance $v_s t$ to the right of O. Consider the wave emitted at time $t'<t$ when the source is at point A, a distance $v_s t'$ to the right of O. At time t, the wavefront of the wave emitted at t' is a sphere of radius $c(t-t')$ centered at A. Consider a point T on the sphere at which the line ST is tangent to the sphere. From the figure, the angle θ is given by

$$\sin\theta = \frac{|\overline{AT}|}{|\overline{AS}|} = \frac{c(t-t')}{v_s(t-t')} = \frac{c}{v_s}. \tag{21}$$

Note that this angle is the same for all $0 \leq t' < t$. Hence the tangent points of the spheres form a cone with vertex at S and the axis along $-\boldsymbol{v}_s$. The angle of the cone from the axis is $\theta = \sin^{-1}(c/v_s)$. This is known as the Mach cone. It is easy to show that $\boldsymbol{\beta}_s(t_r)\cdot\hat{\boldsymbol{n}} = \beta_s \sin\theta = 1$ at T. Thus the shock wave is on the Mach cone. In the case a supersonic aircraft, a thud is heard when the shock wave passes through an observer. This is known as the sonic boom.

Another way to find the surface on which the shock wave lives at time t is to gather all singular points emitted at time $\lambda < t$. It is useful to go through the algebra here since the same analysis will be used to study the shock structure in the circular motion case. Given any direction $\hat{\boldsymbol{m}}$, the wave emitted at time $\lambda < t$ reaches the position

$$\boldsymbol{x}(\hat{\boldsymbol{m}},\lambda) = \boldsymbol{x}_s(\lambda) + c(t-\lambda)\hat{\boldsymbol{m}}. \tag{22}$$

Shocks propagate along the direction in which $\hat{\boldsymbol{m}}\cdot\boldsymbol{\beta}_s = 1$ or $\hat{\boldsymbol{m}}\cdot\hat{\boldsymbol{\beta}}_s = 1/\beta_s$. We can write

$$\hat{\boldsymbol{m}} = \frac{1}{\beta_s}\hat{\boldsymbol{\beta}}_s + \frac{\sqrt{\beta_s^2-1}}{\beta_s}\hat{\boldsymbol{\beta}}_{s\perp} \tag{23}$$

with $\hat{\boldsymbol{\beta}}_{s\perp}\cdot\hat{\boldsymbol{\beta}}_s = 0$. For constant-velocity motion, we can set $\hat{\boldsymbol{\beta}}_s = \hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{\beta}}_{s\perp} = \cos\varphi\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}$ with $\varphi \in [0, 2\pi)$. Then the shock direction is given by

$$\hat{\boldsymbol{m}}_{\rm shock} = \frac{1}{\beta_s}\hat{\boldsymbol{x}} + \frac{\sqrt{\beta_s^2-1}}{\beta_s}(\cos\varphi\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}) \tag{24}$$

and the location of the shock at time t is

$$\begin{aligned} \boldsymbol{x}_{\rm shock}(\varphi,\lambda) &= v_s\lambda\hat{\boldsymbol{x}} + \frac{c(t-\lambda)}{\beta_s}\hat{\boldsymbol{x}} + c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}(\cos\varphi\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}) & (25) \\ &= v_s t\hat{\boldsymbol{x}} + c(t-\lambda)\frac{\beta_s^2-1}{\beta_s}\left[-\hat{\boldsymbol{x}} + \frac{1}{\sqrt{\beta_s^2-1}}(\cos\varphi\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}})\right] & (26) \end{aligned}$$

for $\lambda \in [0, t]$ and $\varphi \in [0, 2\pi)$. It is straightforward to show that the surface is a cone with vertex at $\boldsymbol{x} = \boldsymbol{x}_s(t) = v_s t\hat{\boldsymbol{x}}$. The axis of the cone is along $-\hat{\boldsymbol{x}}$ ($= -\hat{\boldsymbol{v}}_s$) direction and the angle of the cone to the axis is $\theta = \tan^{-1}(1/\sqrt{\beta_s^2-1}) = \sin^{-1}(1/\beta_s)$. This is precisely the Mach cone.

We can see from Figure 1 that no waves can reach outside the Mach cone. Inside the Mach cone, there is a region in which two wavefronts cross, indicating that those are points with two retarded times. The phenomenon can be explained from the equation of the retarded time $c(t-t_r) = |\boldsymbol{x} - \boldsymbol{x}_s(t_r)|$. For constant-velocity motion, $\boldsymbol{x}_s(t_r) = \boldsymbol{v}_s t_r$ and the equation becomes

$$c(t-t_r) = |\boldsymbol{x} - \boldsymbol{v}_s t_r|. \tag{27}$$

It is convenient to consider the transformation $u = c(t - t_r)$ and $\boldsymbol{q} = \boldsymbol{x} - \boldsymbol{v}_s t$. The vector $\boldsymbol{q}$ is the position vector measured from the current source position $\boldsymbol{x}_s(t)$. Equation (27) becomes

$$u = |\boldsymbol{q} + \boldsymbol{\beta}_s u|. \tag{28}$$

Squaring both sides gives

$$u^2 = |\boldsymbol{q} + \boldsymbol{\beta}_s u|^2 = q^2 + \beta_s^2 u^2 + 2\boldsymbol{q} \cdot \boldsymbol{\beta}_s, \tag{29}$$

which can be simplified to

$$(1 - \beta_s^2)u^2 - 2\boldsymbol{q} \cdot \boldsymbol{\beta}_s u - q^2 = 0. \tag{30}$$

The solution to this quadratic equation is

$$c(t - t_r) = u = \frac{\boldsymbol{q} \cdot \boldsymbol{\beta}_s \pm \sqrt{(\boldsymbol{q} \cdot \boldsymbol{\beta}_s)^2 + (1 - \beta_s^2)q^2}}{1 - \beta_s^2} \tag{31}$$

Clearly when $\beta_s < 1$ (i.e. $v_s < c$), there are two solutions: one with $t_r < t$ and the other with $t_r > t$. The one with $t_r < t$ is the solution to Eq. (27). The one with $t_r > t$ is the solution to the equation $c(t_r - t) = |\boldsymbol{x} - \boldsymbol{v}_s t_r|$, which also gives rise to Eq. (29) upon squaring. Hence only the solution with $t_r < t$ is the retarded time. This is consistent with our earlier analysis.

When $\beta_s > 1$, solutions only exist if $(\boldsymbol{q} \cdot \boldsymbol{\beta}_s)^2 + (1 - \beta_s^2)q^2 \geq 0$, which is equivalent to

$$(\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s)^2 \geq \frac{\beta_s^2 - 1}{\beta_s^2} \tag{32}$$

$$\Rightarrow \quad \hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s \leq -\frac{\sqrt{\beta_s^2 - 1}}{\beta_s} \quad \text{or} \quad \hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s \geq \frac{\sqrt{\beta_s^2 - 1}}{\beta_s}. \tag{33}$$

When $\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s \geq \sqrt{\beta_s^2 - 1}/\beta_s$, all two solutions give $t_r > t$. Hence they are not solutions to Eq. (27). Therefore, Equation (27) has solutions only when

$$-\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s \geq \frac{\sqrt{\beta_s^2 - 1}}{\beta_s}. \tag{34}$$

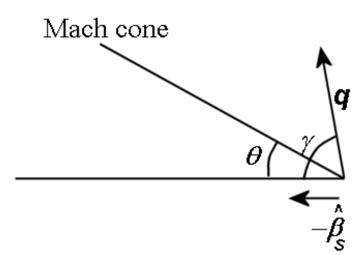

Figure 2: $\boldsymbol{q}$ is inside or on the Mach cone if the angle $\gamma \leq \theta$.

This is actually the condition that $\boldsymbol{q}$ must be inside or on the Mach cone. As shown in Figure 2, the vector $\boldsymbol{q}$ is inside or on the Mach cone only if the angle $\gamma \leq \theta$, where $\theta = \sin^{-1}(1/\beta_s)$ is the Mach angle. Since the cosine function is decreasing for angles between 0 and π, the condition $\gamma \leq \theta$ is equivalent to $\cos\gamma \geq \cos\theta$. Since $\cos\gamma = \hat{\boldsymbol{q}} \cdot (-\hat{\boldsymbol{\beta}}_s)$ and $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{\beta_s^2 - 1}/\beta_s$, the condition is equivalent to $\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{\beta}}_s \leq -\sqrt{\beta_s^2 - 1}/\beta_s$, which is precisely Eq. (34).

Inside the Mach cone, equation (31) gives two valid solutions for the retarded time t_r. We can see from Figure 1 that there is region in which wavefronts cross. One wavefront corresponds to the retarded position $\boldsymbol{x}_s(t_r)$ from the left and the other corresponds to the retarded position from the right. However, there is also a region where there is only one wavefront inside the Mach cone but equation (31) indicates that there are always exactly two values of t_r inside the Mach cone. The reason why we don't see two wavefronts in certain region is also clear from the animation. The wave source is only on at $t \geq 0$ and the one-wavefront region corresponds to the second solution having $t_r < 0$ and it doesn't contribute to wave function because no wave was generated at $t_r < 0$. Mathematically, the vanishing wave function at $t_r < 0$ is imposed by the step function $\Theta(t_r)$ in Eq. (4).

4.2 Circular Motion

In this case, the source moves in a circular trajectory with constant speed v_s. Orient the coordinates so that the source moves in the x-y plane and is on the positive x-axis at $t = 0$. The source position can be written as

$$\boldsymbol{x}_s(t) = r_s(\cos\Omega t\hat{\boldsymbol{x}} + \sin\Omega t\hat{\boldsymbol{y}}), \tag{35}$$

where the radius r_s is constant and $\Omega = v_s/r_s$.

As shown in Figure 3, the shock structure of a circular motion is more complicated than that of the constant-velocity motion. However, the region near the center of the circular motion is relatively simple. Since shock wave propagates out along a cone with angle $\alpha = \cos^{-1}(c/v_s)$ from the velocity vector $\boldsymbol{v}_s$, $\alpha < \pi/2$ and so the shock wave can never reach the center, which is in a direction perpendicular to $\boldsymbol{v}_s$. In fact, it is easy to show that the shock wave can never reach points less than rc/v_s from the motion center.

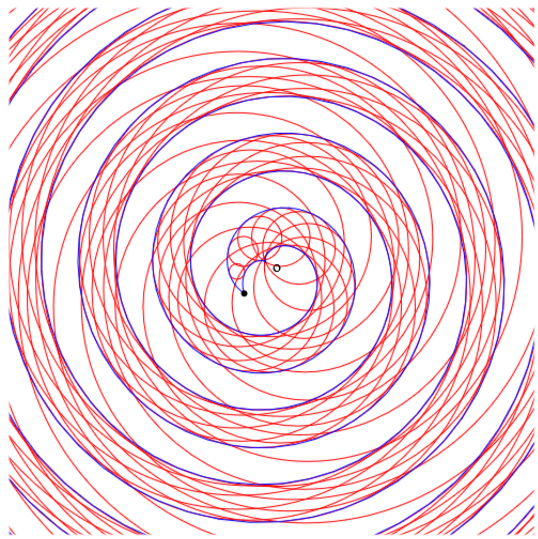

Figure 3: Waves generated by a source (represented by a black dot) moving in a circle with $v_s = 3c$. The center of the circular motion is marked by "o" and blue lines are the location of the shock wave in the x-y plane.

The location of the shock wave at time t can be computed using Eqs. (22) and (23). The vector $\hat{\boldsymbol{\beta}}_s$ is given by

$$\hat{\boldsymbol{\beta}}_s = \frac{\dot{\boldsymbol{x}}_s}{|\dot{\boldsymbol{x}}_s|} = -\sin\Omega t\hat{\boldsymbol{x}} + \cos\Omega t\hat{\boldsymbol{y}}. \tag{36}$$

The vector $\hat{\boldsymbol{\beta}}_{s\perp}$ can be written as a linear combination of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{z}}$ as follows.

$$\hat{\boldsymbol{\beta}}_{s\perp} = \cos\varphi\hat{\boldsymbol{r}} + \sin\varphi\hat{\boldsymbol{z}} \tag{37}$$

with $\varphi \in [0, 2\pi)$ and $\hat{\boldsymbol{r}} = \cos\Omega t\hat{\boldsymbol{x}} + \sin\Omega t\hat{\boldsymbol{y}}$. Hence the location of the shock wave is

$$\begin{aligned}\boldsymbol{x}_{\rm shock}(\varphi,\lambda) \;=\; & \left\{\left[r_s + c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}\cos\varphi\right]\cos\Omega\lambda - \frac{c(t-\lambda)}{\beta_s}\sin\Omega\lambda\right\}\hat{\boldsymbol{x}} \\ & + \left\{\left[r_s + c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}\cos\varphi\right]\sin\Omega\lambda + \frac{c(t-\lambda)}{\beta_s}\sin\Omega\lambda\right\}\hat{\boldsymbol{y}} \\ & + c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}\sin\varphi\,\hat{\boldsymbol{z}}\end{aligned} \tag{38}$$

with $\lambda \in [0, t]$ and $\varphi \in [0, 2\pi)$. The shock surface intersects the x-y plane at $\varphi = 0$ and $\varphi = \pi$, forming two shock lines given by the parametric equations

$$x^{\pm}_{\text{shock}}(\lambda) = \left[r_s \pm c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}\right]\cos\Omega\lambda - \frac{c(t-\lambda)}{\beta_s}\sin\Omega\lambda \tag{39}$$

$$y^{\pm}_{\text{shock}}(\lambda) = \left[r_s \pm c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}\right]\sin\Omega\lambda + \frac{c(t-\lambda)}{\beta_s}\cos\Omega\lambda. \tag{40}$$

with $\lambda \in [0,t]$. The two blue lines in the animation are plotted using these two equations. The equations can be expressed in a compact way using the complex number

$$Z^{\pm}_{\text{shock}}(\lambda) \equiv x^{\pm}_{\text{shock}}(\lambda) + iy^{\pm}_{\text{shock}}(\lambda) = \left[r_s + \frac{c(t-\lambda)}{\beta_s}\left(i \pm \sqrt{\beta_s^2-1}\right)\right]e^{i\Omega\lambda}. \tag{41}$$

The distance of these points from the center is

$$r^{\pm}_{\text{shock}}(\lambda) = |Z^{\pm}_{\text{shock}}(\lambda)| = \sqrt{r_s^2 + c^2(t-\lambda)^2 \pm 2r_s c(t-\lambda)\frac{\sqrt{\beta_s^2-1}}{\beta_s}}. \tag{42}$$

The expression shows that r^2_{shock} is a quadratic function in $c(t-\lambda)$ and has a minimum. The minimum value of r^-_{shock} is r_s/β_s at $\lambda = \lambda_c = t - r_s\sqrt{\beta_s^2-1}/(c\beta_s)$. The minimum value of r^+_{shock} is r_s for $\lambda \in [0,t]$. This calculation confirms that the shock wave can only reach region with radius $r \geq r_s/\beta_s$ from the center.

A closer look at Figure 3 suggests that the inner shock line has a cusp near $r = r_s/\beta_s$. This can be confirmed by noting that

$$\left.\frac{dx^-_{\text{shock}}}{d\lambda}\right|_{\lambda=\lambda_c} = \left.\frac{dy^-_{\text{shock}}}{d\lambda}\right|_{\lambda=\lambda_c} = 0 \tag{43}$$

and both derivatives change sign as λ increases from $\lambda_c - \epsilon$ to $\lambda_c + \epsilon$. This means that both x^-_{shock} and y^-_{shock} reach stationary values at $r = r_s/\beta_s$ and then turn around, creating a cusp. Apart from the cusp, the shock lines trace out two spiral patterns and the patterns rotate as time increases.

Observers located at $r < r_s c/v_s$ do not see a shock wave and the wave frequency oscillates periodically as the source moves. An observer at the center, however, does not see any frequency change. Observers at $r > r_s c/v_s$ will be hit by shock waves repeatedly as the shock lines pass through them. In the case shown in Figure 3, waves generated from multiple retarded times superpose in the region bounded the two shock lines, but only a single retarded time contributes to the wave in the region outside the shock lines. However, things can become complicated when β_s increases. As shown in Figure 4, waves generated from multiple retarded times can superpose at any location with $r > r_s c/v_s$. In addition, the inner shock line Z^-_{shock} cross itself close to the source location $\boldsymbol{x}_s$. So there are places where two shocks collide!

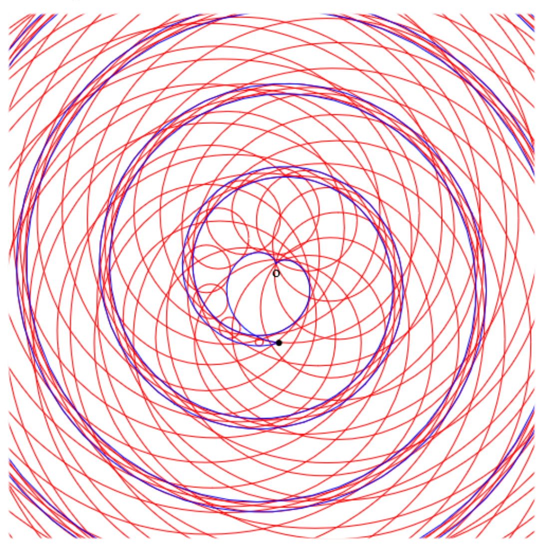

Figure 4: Same as Figure 3 but with $v_s = 5c$.